# Solvability of non-linear integro-differential equation

Mahmoud. M. EL BORAI 1, W. G. EL-SAYED 2 and R. M. BAYOMI 3
Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt
e-mails:1 m_ m_ elborai@yahoo.com, 2 wagdygoma@alexu.edu.eg, 3 rode_angle92@yahoo.com

**AMS (MOS) Subject Classifications:** 34K20, 45D05, 45M05.

**Abstract.** The aim of this paper is to study the existence of weakly continuous and continuous solutions of an integro-differential equation in the space $C[0,1]$.

**Keywords.** b-metric space, weakly (s, r)-contractive multi-valued operator, fixed point theorem.

## 1. Introduction.

Several fixed point theorems such as Banach, Krasnoselskii, Darbo, …etc enable us to prove the existence and uniqueness or existence solutions of many integral equations in different Banach spaces [3,7,10,11].

In this work, we try to establish the sufficient conditions under which we can prove the existence theorem of an integro-differential equation to get weakly continuous solutions in the space $C[0,1]$ by using a fixed point theorem due to Lingjuan Ye and Congcong [16]. Also, we will treat the existence of continuous solutions for the same integro-differential equation in the space $C[0,1]$ by using a fixed point theorem due to Nadler [6]. In the following, we will introduce some definitions and basic theorems that will be used in our paper (see [1- 4, 19-21] ).

**Definition 1.** ($b$-metric space)[5]
Let $X$ be a non-empty set and b $\geq 1$ is a given real number. A function $d: X \times X \to \mathbb{R}^+$ is said to be $b$-metric space if and only if for all $x, y, z \in X$, the following conditions are satisfied:

i) $d(x, y) = 0$ if and only if $x = y$
ii) $d(x, y) = d(y, x)$
iii) $d(x, z) \leq b[d(x, y) + d(y, z)]$

Then the triplet $(X, d, b)$ is called $b$-metric space**.**

**Definition 2.** (complete $b$-metric space)[16]
Let $(X, d, b)$ be a $b$-metric space and $\{x_n\}$ be a sequence of $X$, then we can define the following:

i) $\{x_n\}$ is convergent if there exists an $x$ in $X$ such that for any $\varepsilon < 0$, there exists an $n(\varepsilon) \in N$ such that $n \geq n(\varepsilon)$, $d(x_n, x) < \varepsilon$

ii) $\{x_n\}$ is a Cauchy sequence if for any $\varepsilon < 0$, there exists an $n(\varepsilon) \in N$ such that $m, n \geq n(\varepsilon)$, $d(x_n, x_m) < \varepsilon$

iii) $(X, d)$ is complete if and only if every Cauchy sequence in $X$ is convergent.

**Definition 3.** ((s, r)-contractive multi-valued operator) [16]

Let $(X, d)$ be a metric space, $T: X \rightarrow CB(X)$ be a multi-valued operator. If there exist constants s, r with $r \in [0, 1], s \geq r$ such that for all $x, y \in X$

$$d(y, Tx) \leq Ksd(x, y) \Rightarrow H(Tx, Ty) \leq rM_T(x, y)$$

Where

$$M_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2K}\right\}$$

Then $T$ is called a $(s, r)$-contractive multi-valued operator.

**Definition 4.** (weakly (s, r)-contractive multi-valued operator) [16]

Let $(X, d)$ be a metric space, $T: X \rightarrow CB(X)$ be a multi-valued operator. If there exist $r \in [0, 1], s \geq r, L \geq 0$ such that for all $x, y \in X$

$$d(y, Tx) \leq Ksd(x, y) \Rightarrow H(Tx, Ty) \leq rM_T(x, y) + L \min\{d(x, y), d(y, Tx)\}$$

Where

$$M_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2K}\right\}$$

Then $T$ is called a weakly $(s, r)$-contractive multi-valued operator.

**Remark.** when $L = 0$, the above definition reduces to definition 3.

**Definition 5.** (Hausdorff b-metric) [17]

For $A, B \in CB(X)$, define the function $H: CB(X) \times CB(X) \rightarrow \mathbb{R}^+$ by

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}$$

where

$$\delta(A, B) = \sup\{d(a, B), a \in A\}, \qquad \delta(B, A) = \sup\{d(b, A), b \in B\}$$

with

$$d(a, C) = \inf\{d(a, x), x \in C\}$$

$H$ is called the Hausdorff $b$-metric induced by the $b$-metric $d$

We recall the following properties from [5, 6]

**Theorem 1**

Let $(X, d, s)$ be a $b$-metric space. For any $A, B, C \in CB(X)$, [$CB(X)$ denotes the family of non-empty, closed and bounded subsets of $X$] and any $a, b, c \in X$, we have the following:

1) $d(x, B) \leq d(x, b)$ for any $b \in B$
2) $\delta(A, B) \leq H(A, B)$

ISSN 2229-5518

3) $d(x,B) \leq H(A,B)$ for any $a \in A$
4) $H(A,A) = 0$
5) $H(A,B) = H(B,A)$
6) $H(A,C) \leq s[H(A,B) + H(B,C)]$
7) $d(x,A) \leq s[d(x,y) + d(y,A)]$

**Theorem 2**

Let $(X,d,s)$ be a $b$-metric space. For any $A,B \in CB(X)$. Then for each $h > 1$ and for each $a > A$, there exists $b(a) \in B$ such that $d\big(a, b(a)\big) \leq hH(A,B)$

**Theorem 3**

Let $(X,d,s)$ be a $b$-metric space. For any $A \in CB(X)$ and $x \in X$, we have $d(x,A) = 0 \;\; iff \;\; x \in \overline{\mathrm{A}} = \mathrm{A}$, where $\overline{\mathrm{A}}$ denotes the closure of the set A.

**Theorem 4**

Let $(X,d,s)$ be a $b$-metric space and let $\{x_n\}$ be a sequence in $X$. If $\lim_{n\to\infty^+} x_n = y$ and $\lim_{n\to\infty^+} x_n = z$, then $y = z$

**Theorem 5**

Let $(X,d,s)$ be a $b$-metric space and let $\{x_n\}$ be a sequence in $X$ such that

$$d(x_n, x_{n+1}) \leq \lambda\, d(x_{n-1}, x_n)$$

For some $\lambda \in (0, s^{-1})$ and each $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence in $X$



## 2. Main results

Consider the integro-differential equation

$$x(t) = g(t) + \int_0^1 k(t,s)\, f\big(s, x'(s)\big)\, ds \;, \qquad t \in [0,1] \qquad (1)$$

Now, we will create an integral equation which equivalent to the integro-differential equation (1), by differentiating equation (1) with respect to $t$, hence we get

$$x'(t) = g'(t) + \int_0^1 k'(t,s)\, f\big(s, x'(s)\big)\, ds \;, \quad t \in [0,1]$$

Assume that

$$x'(t) = y(t), \quad g'(t) = h(t), \quad k'(t,s) = v(t,s)$$

Then the last integral equation becomes

$$y(t) = h(t) + \int_0^1 v(t,s)\, f\big(s, y(s)\big)\, ds \,, \quad t \in [0,1] \quad (2)$$

So, the solution of our integro-differential equation (1) can be determined if the solution of the equivalent integral equation (2) is obtained.

First, we will treat a weakly continuous solution of our integro-differential equation (1) in the space $C[0,1]$; the space of all continuous functions defined on an interval $I = [0,1]$, with metric

$$d(x, y) = \max_{t\in[0,1]} |x(t) - y(t)|^2, \qquad \forall x, y \in C[0,1]$$

In this case, we see that the space $(C, d)$ is a complete $b$-metric space with $K = 2$.

We will use in this section the following fixed point theorem due to Lingjuan Ye and Congcong Shen [16].

**Theorem 2.1**

Let $(X, d)$ be a complete $b$-metric space and let $T: X \to CB(X)$ be a weakly (s, r)-contractive operator with $r < \min\left\{\frac{1}{K}, s\right\}$. Then $T$ has fixed point.

Let $X = C[0,1]$ and define the operator

$$T: X = C[0,1] \to CB(X)$$

where $CB(X)$ denotes the space of all nonempty, closed and bounded subsets in $X$, as:

$$Ty(t) = h(t) + \int_0^1 v(t,s)\, f\big(s, y(s)\big)\, ds$$

Now, we will consider some assumptions under which our existence theorem can be proved.

Assume that:

(i) $h \in C(I)$

(ii) $v = v(t,s): I \times I \to \mathbb{R}$ is continuous with respect to its two variables $t, s$ such that $\int_0^1 |v(t,s)|^2\, ds < \alpha$ , $0 \le \alpha \le 1$

(iii) $f = f\big(t, x(t)\big) : I \times \mathbb{R} \to \mathbb{R}$ is continuous such that

$$|f(t, y_1) - f(t, y_2)|^2 \le r|y_1 - y_2|^2 , \quad r = min\left\{\frac{1}{2}, s\right\} \ , s > 0$$

(iv) For each $x \in C(I, \mathbb{R})$, the multivalued operator $f: I \times \mathbb{R} \to K_{cv}(\mathbb{R})$ is such that $f\big(s, x(s)\big)$ is lower semicontinuous in $I \times I$

Now, we have the main result in the following theorem

**Theorem 2.2**

If the assumptions (i) – (iv) are satisfied, then there exists at least a weakly continuous solution of the integral equation (3.2) in the space $C[0,1]$.

**Proof**

First, we can deduce from the assumption (ii) that the integral operator

$$(Vx)(t) = \int_0^1 v(t,s)x(s)ds$$

Will be continuous and transforms the space $C[0,1]$ into itself [14], also due to assumption (iii) we deduce that the superposition generated by the function $f$ transforms space $C[0,1]$ into itself [14], so using our assumptions, we see that the operator $T$ transforms the space $X = C[0,1]$ into the space $CB(X)$, i.e.

$$T: X = C[0,1] \to CB(X)$$

Next, since

$$d(Ty_1, Ty_2) = \max_{|y_1|,|y_2|<\delta} \left| \int_0^1 v(t,s)\, f\big(s, y_1(s)\big) - v(t,s)\, f\big(s, y_2(s)\big)\, ds \right|^2$$

$$\leq \max_{|y_1|,|y_2|<\delta} \int_0^1 \left| v(t,s) \left[ f\big(s, y_1(s)\big) - f\big(s, y_2(s)\big) \right] \right|^2 ds$$

$$\leq \alpha \max_{|y_1|,|y_2|<\delta} \int_0^1 \left| \left[ f\big(s, y_1(s)\big) - f\big(s, y_2(s)\big) \right] \right|^2 ds$$

$$\leq \alpha \max_{|y_1|,|y_2|<\delta} \int_0^1 r |y_1(s) - y_2(s)|^2\, ds$$

$$\leq \alpha\, r \max_{|y_1|,|y_2|<\delta} |y_1(t) - y_2(t)|^2$$

$$= \alpha\, r\ \mathrm{d}(y_1(t), y_2(t))$$

$$\leq \alpha\, r\, M_T\big(y_1(t), y_2(t)\big) + \lim\left\{ d\big(y_1(t), y_2(t)\big), d\big(y_2(t), Ty_1(t)\big) \right\}$$

Where $M_T(y_1, y_2) = \max\left\{ d(y_1, y_2), d(y_1, Ty_1), d(y_2, Ty_2), \frac{d(y_1, Ty_2) + d(y_2, Ty_1)}{4} \right\}$

Then T satisfies the conditions of theorem (2.1) and T has a fixed point ■

Next, we will investigate a solution of our integro-differential equation (1) in the space $C(I, \mathbb{R})$ of all continuous real functions defined on an interval $I \subset \mathbb{R}$, with norm

$$\|x\| = \sup_{t \in I} \|x(t)\|$$

In this section, we will use the following fixed point theorem due to Nadler [6].

ISSN 2229-5518

**Theorem 2.3**

Let $(X,d,s)$ be a complete $b$-metric space and let $T: X \to CB(X)$. Assume that there exists $k \in (0,1)$ such that $sH(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$, then $T$ has a fixed point.

Now, we will consider following assumptions under which our theorem will be proved. Assume that:

(i) $h \in C(I,\mathbb{R})$

(ii) For each $x \in C(I,\mathbb{R})$, the multivalued operator $f: I \times \mathbb{R} \to K_{cv}(\mathbb{R})$ is such that $f\big(s,x(s)\big)$ is lower semicontinuous in $I \times I$

(iii) The function $v = v(t,s): I \times I \to \mathbb{R}$ is continuous with respect to its two variables $t,s$ and there exists a constant $k \in (0,1)$, such that

$$\sup_{t\in I} \int_0^1 v(t,s)ds \leq \sqrt{\frac{k}{2}}$$

(iv) there exists $l(t) \in L^1(I)$, for each $t \in I$ such that

$H\big(f(s,x), f(s,y)\big) \leq l(s)|x(s) - y(s)|$, for all $s \in I$ and for all $x,y \in \mathbb{R}$

Hence, we have the following theorem

**Theorem 2.4**

If the conditions (i) – (iv) are satisfied, then equation (3.2) has at least one solution in $C(I,\mathbb{R})$

**Proof**

Let us define the multivalued operator

$$T: C(I,\mathbb{R}) \longrightarrow CL(I,\mathbb{R})$$

Where $CL(I,\mathbb{R})$ denotes the space of all non-empty and closed functions defined on an interval $I \subset \mathbb{R}$, as

$$Tx(t) = \left\{u \in C(I,\mathbb{R}) : u(t) \in h(t) + \int_0^1 v(t,s)\, f\big(s,x(s)\big)\, ds\,, t \in I\right\}$$

for each $x \in C(I,\mathbb{R})$.

Let $x \in C(I,\mathbb{R})$ and $f_x(s) = f(s,x(s))$, $s \in I$, for the multivalued operator $f_x: I \to K_{cv}(\mathbb{R})$, by Michael's selection theorem, we get that there exists a continuous operator $g_x: I \to \mathbb{R}$, such that $g_x(s) \in f_x(s)$, for all $t \in I$.

This implies that $h(t) + \int_0^1 v(t,s) f_x(s)\, ds \in Tx$ and so $Tx$ is a non-empty set. It is an easy matter to show that $T$ is closed.

Let $x_n \longrightarrow x$, $Tx_n \longrightarrow y$, we want to prove that $Tx = y$

Since

$$Tx_n = g(t) + \int_0^1 k(t,s)\, f\big(s, x_n(s)\big)ds$$

then as $n \to \infty$, we have

$$Tx_n \to g(t) + \int_0^1 k(t,s)\, f\big(s, x(s)\big)ds \ = Tx$$

Hence, $y = Tx$ and so $T$ is closed.

Next, we show that the multivalued operator $T$ satisfies all the hypotheses of theorem (2.3)

Let $x, y \in C(I, \mathbb{R})$ be such that $w_1 \in Tx$, then there exists $g_x(s)$ with $s \in I$ such that $w_1(t) \in h(t) + \int_0^1 v(t,s)\, g_x(s)ds, t \in I$.

On the other hand, by hypothesis (iii), we get

$$H\Big(f\big(s, x(s)\big), f\big(s, y(s)\big)\Big) \leq l(s)|x(s) - y(s)|, \text{ for all} \in I\ , x, y \in \mathbb{R}$$

Consequently, there exists $w_2 \in f_y(s)$ such that

$$|g_x(s) - w_2(s)| \leq l(s)|x(s) - y(s)| \text{ for all } s \in I$$

Now, we can consider the multivalued operator $S$ defined by

$S(s) = \ f_y(s) \cap \{u \in \mathbb{R} :\ |f_x(s) - u| \leq l(s)|x(s) - y(s)|\}, s \in I$

Taking into the account the fact the multivalued operator $f$ is lower semicontinuous, it follows that there exists a continuous operator $f_y: I \to \mathbb{R}$ such that $f_y(s) \in S(s), s \in I$.

We have

$$z(t) = h(t) + \int_0^1 v(t,s)\, f_y(s)ds \in h(t) + \int_0^1 v(t,s)\, f(s, y(s))ds$$

and

$$\begin{aligned}
|w_1(t) - z(t)|^2 &\leq \left(\int_0^1 v(t,s)\, \big|f_x(s) - f_y(s)\big| ds\right)^2 \\
&\leq \left(\int_0^1 v(t,s)\, |x(s) - y(s)| ds\right)^2 \\
&\leq \left(\int_0^1 v(t,s) \sqrt{\big(x(s) - y(s)\big)^2} ds\right)^2 \\
&\leq \left(\int_0^1 v(t,s) \sqrt{\|(x-y)^2\|_\infty} ds\right)^2 \\
&\leq \|(x-y)^2\|_\infty \left(\int_0^1 v(t,s)\, ds\right)^2 \\
&\leq \|(x-y)^2\|_\infty \left(\sqrt{\frac{k}{2}}\right)^2
\end{aligned}$$

So, we have

$$d(v, z) \leq \frac{k}{2} d(x, y)$$

Then

$$2H(Tx,Ty) \leq kd(x,y) \,, \qquad \text{for all } x,y \in C(I,\mathbb{R})$$

Thus, all the conditions of theorem (2.3) are satisfied, and then equation (2) has a fixed point. ■

## 3. References

[1] Aydi, H, Bota, M-F, Karapinar, E, Mitrović, S: A fixed point theorem for set-valued quasicontractions in $b$-metric spaces. Fixed Point Theory Appl. 2012, Article ID 88 (2012)

[2] Bakhtin, IA: The contraction mapping principle in quasimetric spaces. In: Functional Analysis, vol. 30, pp. 26-37.Ul'yanovsk. Gos. Ped. Inst., Ul'yanovsk (1989) (in Russian)

[3] S. Banach, Sur les operat6ttg btions dans les ensembles abstraits et leures applications aux equations integrales, Fundam. Math., 3 (1922), 133–181. 1

[4] M. Boriceanu, M. Bota, A. Petruşel, Multivalued fractals in $b$-metric spaces, Cent. Eur. J. Math., 8 (2010), 367–377. 1,1.3

[5] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostraviensis, 1 (1993), 5–11. 1, 1.1

[6] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Semin. Mat. Fis. Univ. Modena, 46 (1998), 263–276. 1, 1.7

[7] M. M.El-Borai,W. G.El-Sayed and A. M.Moter, Continuous Solutions of a Quadratic Integral Equation, Inter.J.Life Science and Math.(IJLSM), Vol.2 (5)-4, (2015), 21-30.

[8] M. M. El-Borai, Khairia El-Said El-Nadi, Integrated semi groups and Cauchy problem for some fractional abstract differential equations. Life science Journal, 2013, 10(3), 793-795.

[9] M. M. El-Borai, Khairia El-Said El-Nadi and Eman G. El-Akabawi, On some integro-differential equations of fractional orders, the inter.l J. of Contemporary Mathematics, Vol. 1. No 15, (2006), pp. 719-726.

[10] M. M. El-Borai, On solvability of an inverse fractional abstract Cauchy problrm, Inter. J.l of research and reviews in applied science, Vol. 4. September, 2010, No. 4, pp. 411-416.

[11] M. M. El-Borai, W. G. El-Sayed and Fayez N. Ghaffoori. On the solvability of non-linear integral functional equation, (IJM) vo.34 No.1 june 2016.

[12] M. M. El-Borai, the fundamental solution for fractional evolution equations of parabolic type, J. of Appl. Math. Stochastic Analysis (JAMSA), (2004),199-211.

[13] W. G. El-Sayed, Monotonic solution of functional integral equation, J-Egypt. Math. Soc.Vol. 14(2)(2006) pp 235-241.

[14] W. G. El-Sayed, B. Rzepka, Non-decreasing solutions of quadratic integral equation of Urysohn type, comp. Math. Appl. 51(2006) 1065-1074.

[15] W. G. El-Sayed, Non-linear functional integral equations of convolution type, Portugaliae Mathematica 54 (4)(1997) 449-456.

[16] Lingjuan Ye, Congcong Shen. J.Non-linear Sci. Appl.,11 (2018).

[17] Monica Cosentino, Mohamed Jleli, Bassem Samet and Calogero Vetro. Cosentino et al.

ISSN 2229-5518
Fixed point theorem and Application (2015).
[18] S. B. Nadler, Multi-valued contraction mappings, Pacific J. Math., 30 (1969), 475–488. 1
[19] J. R. Roshan, V. Parvaneh, Z. Kadelburg, Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces, J. Nonlinear Sci. Appl., 7 (2014), 229–245. 1, 1.4, 2.4
[20] S. L. Singh, S. Czerwik, K. Kr´ ol, A. Singh, Coincidences and fixed points of hybrid contractions, Tamsui Oxf. J. Math.Sci., 24 (2008), 401–416. 1
[21] L. Wang, The fixed point method for intuitionistic fuzzy stability of a quadratic functional equation, Fixed Point Theory. Appl., 2010 (2010), 7 pages. 1